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Dunajski generalization of the second heavenly equation: dressing method and the hierarchy

L V Bogdanov[1], **V S Dryuma**[2] **and S V Manakov**[1]

[1] L D Landau ITP, Kosygin str 2, Moscow 119334, Russia
[2] IMI AS RM, Academy str 5, MD2001, Kishinev, Moldova

E-mail: leonid@landau.ac.ru

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Abstract
Dunajski generalization of the second heavenly equation is studied. A dressing scheme applicable to the Dunajski equation is developed; an example of constructing solutions in terms of implicit functions is considered. A Dunajski equation hierarchy is described, and its Lax–Sato form is presented. The Dunajski equation hierarchy is characterized by the conservation of a three-dimensional volume form, in which a spectral variable is taken into account.

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1. Dunajski equation

In this work we study an integrable model proposed by Dunajski [1]. This model is a representative of the class of integrable systems arising in the context of complex relativity [2–7]. It is closely connected to the Plebański second heavenly equation [2] and in some sense generalizes it. The starting point is a (2,2) signature metric in the canonical Plebański form

$$g = \mathrm{d}w\,\mathrm{d}x + \mathrm{d}z\,\mathrm{d}y - \Theta_{xx}\,\mathrm{d}z^2 - \Theta_{yy}\,\mathrm{d}w^2 + 2\Theta_{xy}\,\mathrm{d}w\,\mathrm{d}z. \tag{1}$$

Vacuum Einstein equations and conformal anti-self-duality (ASD) condition for this metric lead to the celebrated Plebański second heavenly equation [2],

$$\Theta_{wx} + \Theta_{zy} + \Theta_{xx}\Theta_{yy} - \Theta_{xy}^2 = 0.$$

This equation is one of the most known integrable models of 4D self-dual gravity.

Dunajski has demonstrated that omitting one of the conditions (Einstein equations) and imposing only conformal anti-self-duality, one still gets an integrable system, which can be written as

$$\Theta_{wx} + \Theta_{zy} + \Theta_{xx}\Theta_{yy} - \Theta_{xy}^2 = f, \tag{2}$$

$$f_{xw} + f_{yz} + \Theta_{yy} f_{xx} + \Theta_{xx} f_{yy} - 2\Theta_{xy} f_{xy} = 0. \tag{3}$$

Equations (2), (3) represent a compatibility condition for the linear system $L_0\Psi = L_1\Psi = 0$, where $\Psi = \Psi(w, z, x, y, \lambda)$ and

$$\begin{aligned} L_0 &= (\partial_w - \Theta_{xy}\partial_y + \Theta_{yy}\partial_x) - \lambda\partial_y + f_y\partial_\lambda, \\ L_1 &= (\partial_z + \Theta_{xx}\partial_y - \Theta_{xy}\partial_x) + \lambda\partial_x - f_x\partial_\lambda. \end{aligned} \tag{4}$$

A similar model, corresponding to the first heavenly equation, was introduced in [8]. It was demonstrated that a volume-preserving Riemann–Hilbert problem is connected with the model [9].

In the context of integrable systems, the place of Dunajski equation is in the class of integrable equations connected with the commutation of vector fields. Important and extensively studied representatives of this class are dispersionless integrable equations, and also heavenly equations and hyper-Kähler hierarchies [10, 11]. The Dunajski equation combines the properties of both cases. Similar to dispersionless integrable systems (and opposite to heavenly type equations), it contains a derivative over the spectral parameter in the Lax pair (4).

The main reason of our interest in the Dunajski equation is that it generalizes both dispersionless KP and Plebański second heavenly equation. The study of this model helps to understand the features of general hierarchy connected with the commutation of vector fields. This model is also useful in the theory of ordinary differential equations. Connections of the Dunajski equation with the theory of ordinary differential equations were considered in [12].

In this work we study the Dunajski equation (2), (3) as an integrable system. The dressing method based on the nonlinear volume-preserving Riemann–Hilbert problem is developed and used to construct special solutions. The Dunajski equation hierarchy is described in terms of a volume form; its Lax–Sato equations are presented. Compatibility of the flows of the hierarchy is proved. A preliminary sketch of the results was given in [13].

2. Dressing scheme

Let us consider the nonlinear vector Riemann problem of the form

$$\mathbf{\Psi}_+ = \mathbf{F}(\mathbf{\Psi}_-), \tag{5}$$

where $\mathbf{\Psi}_+$, $\mathbf{\Psi}_-$ denote the boundary values of the N-component vector function on the sides of some oriented curve γ in the complex plane of the variable λ. The problem is to find the function analytic outside the curve with some fixed behavior at infinity (normalization) which satisfies (5). The problem of this type was used by Takasaki [9–11], who stressed its connection to the Penrose nonlinear graviton construction.

Problem (5) is connected with a class of integrable equations, which can be represented as a commutation relation for vector fields containing a derivative on the spectral variable (see [14–16]). We give a sketch of the dressing scheme associated with this problem.

The more specific setting relevant for the Dunajski equation is the following. We consider a three-component Riemann problem (5)

$$\begin{aligned} \Psi_+^0 &= F^0(\Psi_-^0, \Psi_-^1, \Psi_-^2), \\ \Psi_+^1 &= F^1(\Psi_-^0, \Psi_-^1, \Psi_-^2), \\ \Psi_+^2 &= F^2(\Psi_-^0, \Psi_-^1, \Psi_-^2) \end{aligned} \tag{6}$$

for the functions

$$\begin{aligned}
&\Psi^0 \to \lambda + O\left(\frac{1}{\lambda}\right),\\
&\Psi^1 \to -\lambda z + x + O\left(\frac{1}{\lambda}\right),\\
&\Psi^2 \to \lambda w + y + O\left(\frac{1}{\lambda}\right), \qquad \lambda \to \infty,
\end{aligned} \tag{7}$$

where x, y, w, z are the variables of the Dunajski equation (times). We suggest that for a given F the solution of the problem (6) for the functions (7) exists and is unique (at least locally in x, y, w, z).

To establish a correspondence with the twistor construction, one should consider functions Ψ as coordinates on the twistor space, and the curve γ should be closed and belong to the overlap of two neighborhoods of zero and infinity (see [1, 9–11] and references therein). A correspondence with the twistor construction makes it possible to use the results on the existence of solutions obtained within the twistor method [1, 4].

Let us consider a linearized problem

$$\delta\Psi_+^i = \sum_{j=0,1,2} F^i_{,j}\delta\Psi_-^j.$$

The linear space of solutions of this problem is spanned by the infinite basis of functions $\lambda^n\boldsymbol{\Psi}_x, \lambda^n\boldsymbol{\Psi}_y, \lambda^n\boldsymbol{\Psi}_\lambda$, where n is an integer, $0 \leqslant n < \infty$, which can be multiplied by an arbitrary function of times (entering as parameters).

The presence of $\lambda^n\boldsymbol{\Psi}_\lambda$ in the basis is the main difference between the dressing schemes for the heavenly equation [14, 17] and Dunajski equation.

Expanding the functions $\boldsymbol{\Psi}_z, \boldsymbol{\Psi}_w$ into the basis, we obtain linear equations

$$\begin{aligned}
&((\partial_w + u_y\partial_y + v_y\partial_x) - \lambda\partial_y + f_y\partial_\lambda)\boldsymbol{\Psi} = 0,\\
&((\partial_z - u_x\partial_y - v_x\partial_x) + \lambda\partial_x - f_x\partial_\lambda)\boldsymbol{\Psi} = 0,
\end{aligned} \tag{8}$$

where u, v, f can be expressed through the coefficients of the expansion of Ψ^0, Ψ^1, Ψ^2 at $\lambda = \infty$,

$$\begin{aligned}
&u = \Psi^2_1 - w\Psi^0_1, \qquad v = \Psi^1_1 + z\Psi^0_1, \qquad f = \Psi^0_1,\\
&\Psi^0 = \lambda + \sum_{n=1}^{\infty}\frac{\Psi^0_n}{\lambda^n}, \qquad \Psi^1 = -z\lambda + x + \sum_{n=1}^{\infty}\frac{\Psi^1_n}{\lambda^n},\\
&\Psi^2 = w\lambda + y + \sum_{n=1}^{\infty}\frac{\Psi^2_n}{\lambda^n}, \qquad \lambda = \infty.
\end{aligned} \tag{9}$$

The compatibility condition for equations (8) represents a closed system of equations for three functions u, v, f, and solutions to this system can be found using the problem (6).

To get a Lax pair for the Dunajski equation from linear equations (8), we should consider the reduction $v_x = -u_y$, then we can introduce a potential Θ,

$$v = \Theta_y, \qquad u = -\Theta_x. \tag{10}$$

Proposition 1. *Sufficient condition to provide the reduction*

$$v_x = -u_y$$

in terms of the Riemann problem (6) is

$$\det F^i_{,j} = 1. \tag{11}$$

Proof. Condition (11) implies that

$$\mathrm{d}\Psi^0_+ \wedge \mathrm{d}\Psi^1_+ \wedge \mathrm{d}\Psi^2_+ = \mathrm{d}\Psi^0_- \wedge \mathrm{d}\Psi^1_- \wedge \mathrm{d}\Psi^2_-,$$

and thus the form

$$\Omega = \mathrm{d}\Psi^0 \wedge \mathrm{d}\Psi^1 \wedge \mathrm{d}\Psi^2$$

is analytic in the complex plane. Then the determinant of the matrix

$$J = \begin{pmatrix} \Psi^0_\lambda & \Psi^1_\lambda & \Psi^2_\lambda \\ \Psi^0_x & \Psi^1_x & \Psi^2_x \\ \Psi^0_y & \Psi^1_y & \Psi^2_y \end{pmatrix} \tag{12}$$

is also analytic, and, considering the behavior of this determinant at $\lambda = \infty$, we come to the conclusion that

$$\det J = 1.$$

Calculating the coefficient of the expansion of det J at $\lambda = \infty$ corresponding to λ^{-1}, we get

$$\Psi^1_{1x} + z\Psi^0_{1x} + \Psi^2_{1y} - w\Psi^0_{1y} = 0,$$

then, according to (9),

$$v_x = -u_y.$$

□

2.1. An example

Now let us consider a simple example of constructing a solution to the Dunajski equation using problem (6). We introduce the problem of the form

$$\Psi^1_+ = \Psi^1_-, \tag{13}$$

$$\Psi^2_+ = \Psi^2_- \exp(-\mathrm{i}F(\Psi^2_-\Psi^0_-, \Psi^1_-)), \tag{14}$$

$$\Psi^0_+ = \Psi^0_- \exp(\mathrm{i}F(\Psi^2_-\Psi^0_-, \Psi^1_-)), \tag{15}$$

where F is an arbitrary function of two variables.

It is easy to check that the reduction condition (11) is indeed satisfied in this case. Equation (13) implies that $\Psi^1 = -\lambda z + x$. Substituting this solution to linear equations (8) (or using expression (9)), we obtain $v = zf$.

The second important property of the problems (13)–(15) we use is that $\Psi^2_-\Psi^0_- = \Psi^2_+\Psi^0_+$, thus the function $\phi = \Psi^2\Psi^0$ is analytic. Then, taking into account the behavior of functions at infinity (7), we come to the conclusion that ϕ is a polynomial of the form

$$\phi = \Psi^2\Psi^0 = \lambda^2 w + \lambda y + 2fw + u. \tag{16}$$

Equation (15) now reads

$$\Psi^0_+ = \Psi^0_- \exp(\mathrm{i}F(\phi, -\lambda z + x)).$$

The solution to this equation (considered as a standard scalar Riemann–Hilbert problem, see, e.g., [18]) looks like

$$\Psi^0 = \lambda \exp\left(\frac{1}{2\pi}\int_\gamma \frac{\mathrm{d}\lambda'}{\lambda - \lambda'} F(\phi(\lambda'), -\lambda' z + x)\right).$$

Considering the expansion of this expression in λ, we obtain the equations

$$\frac{1}{2\pi}\int_\gamma F(\phi(\lambda), -\lambda z + x)\,\mathrm{d}\lambda = 0, \tag{17}$$

$$\frac{1}{2\pi \mathrm{i}}\int_\gamma \lambda F(\phi(\lambda), -\lambda z + x)\,\mathrm{d}\lambda = f. \tag{18}$$

Taking into account expression (16), we come to the conclusion that these equations define the functions u, f as implicit functions. A solution to the Dunajski equation is then defined by the relations

$$\Theta_x = -u, \qquad \Theta_y = zf.$$

Thus we have obtained a solution to the Dunajski equation, depending on the arbitrary function of two variables, in terms of implicit functions.

To simplify the example and exclude integration with respect to λ, it is possible to consider the function F of the form

$$F(\phi, \Psi^1) = \sum_i \frac{f_i(\phi)}{\Psi^1 - c_i} = \sum_i \frac{f_i(\phi)}{-\lambda z + x - c_i},$$

where f_i are some analytic functions and c_i some constants. Then, performing integration in equations (17), (18) (considering γ as a small circle around infinity) we obtain

$$\sum_i f_i\left(\phi\left(\frac{x - c_i}{z}\right)\right) = 0, \tag{19}$$

$$\sum_i \frac{x - c_i}{z} f_i\left(\phi\left(\frac{x - c_i}{z}\right)\right) = zf, \tag{20}$$

and implicit equations for u, f (17), (18) (compare (16)) simplify considerably.

Functional dependence on the function of two variables F indicates that the solution we have constructed corresponds to some (2+1)-dimensional reduction of the Dunajski equation. It is possible to find the reduced equations explicitly using the fact that linear equations (4) have analytic (polynomial) solutions ϕ and $-\lambda z + x$. Substituting these solutions to linear problems (4) and using relations (10), we come to the conclusion that the solution of the Dunajski equation we constructed also satisfies a pair of (2+1)-dimensional equations

$$(\partial_w - \Theta_{xy}\partial_y + \Theta_{yy}\partial_x)\left(2\frac{w}{z}\Theta_y - \Theta_x\right) + \frac{y}{z}\Theta_{yy} = 0, \tag{21}$$

$$(\partial_z + \Theta_{xx}\partial_y - \Theta_{xy}\partial_x)\left(2\frac{w}{z}\Theta_y - \Theta_x\right) - \frac{y}{z}\Theta_{xy} = 0. \tag{22}$$

Independent variables for the first equation are (w, x, y), and it contains z as a parameter, for the second (z, x, y) and it contains w as a parameter. A common solution to these two equations gives a solution to the Dunajski equation, where f is defined by the relation $zf = \Theta_y$. This common solution is defined by Cauchy data depending on two variables (x, y), thus the functional freedom in the space of solutions satisfying reduction (21), (22) is described by the function of two variables.

3. Dunajski equation hierarchy

The framework developed here is closely connected with the framework of the hyper-Kähler hierarchy developed by Takasaki [10, 11], see also [17, 19]. Though there are some essential differences (the volume form is used instead of the symplectic form, the spectral variable is included to the form), the technique and ideas of the proofs are very similar. We should also mention an integrable generalization of the first heavenly equation proposed by Park [8] and studied by Takasaki [9], connected with volume-preserving diffeomorphisms. However, the hierarchy for this model was not considered. Later a kind of volume preserving hierarchy was introduced in [20], but the hierarchy described in our work does not coincide with it.

To define the Dunajski equation hierarchy, we consider three formal series, depending on two infinite sets of additional variables (times)

$$\Psi^0 = \lambda + \sum_{n=1}^{\infty} \Psi_n^0(\mathbf{t}^1, \mathbf{t}^2)\lambda^{-n}, \tag{23}$$

$$\Psi^1 = \sum_{n=0}^{\infty} t_n^1(\Psi^0)^n + \sum_{n=1}^{\infty} \Psi_n^1(\mathbf{t}^1, \mathbf{t}^2)(\Psi^0)^{-n} \tag{24}$$

$$\Psi^2 = \sum_{n=0}^{\infty} t_n^2(\Psi^0)^n + \sum_{n=1}^{\infty} \Psi_n^2(\mathbf{t}^1, \mathbf{t}^2)(\Psi^0)^{-n}, \tag{25}$$

where $\mathbf{t}^1 = \left(t_0^1, \ldots, t_n^1, \ldots\right)$, $\mathbf{t}^2 = \left(t_0^2, \ldots, t_n^2, \ldots\right)$. We denote $x = t_0^1$, $y = t_0^2$, $\boldsymbol{\Psi} = \left(\begin{smallmatrix}\Psi^0\\ \Psi^1\\ \Psi^2\end{smallmatrix}\right)$, $\partial_n^1 = \frac{\partial}{\partial t_n^1}$, $\partial_n^2 = \frac{\partial}{\partial t_n^2}$ and introduce the projectors $\left(\sum_{-\infty}^{\infty} u_n\lambda^n\right)_+ = \sum_{n=0}^{\infty} u_n\lambda^n$, $\left(\sum_{-\infty}^{\infty} u_n\lambda^n\right)_- = \sum_{-\infty}^{n=-1} u_n\lambda^n$.

The Dunajski equation hierarchy is defined by the relation

$$(\mathrm{d}\Psi^0 \wedge \mathrm{d}\Psi^1 \wedge \mathrm{d}\Psi^2)_- = 0, \tag{26}$$

where the differential includes both times and a spectral variable

$$\mathrm{d}f = \sum_{n=0}^{\infty} \partial_n^1 f\,\mathrm{d}t_n^1 + \sum_{n=0}^{\infty} \partial_n^2 f\,\mathrm{d}t_n^2 + \partial_\lambda f\,\mathrm{d}\lambda.$$

This is a crucial difference with the heavenly equation hierarchy, where only the times are taken into account. Relation (26) plays a role similar to the role of the famous Hirota bilinear identity for the KP hierarchy. This relation is equivalent to the Lax–Sato form of the Dunajski equation hierarchy.

Proposition 2. *Relation (26) is equivalent to the set of equations*

$$\partial_n^1 \boldsymbol{\Psi} = \sum_{i=0,1,2} \left(J_{1i}^{-1}(\Psi^0)^n\right)_+ \partial_i \boldsymbol{\Psi}, \tag{27}$$

$$\partial_n^2 \boldsymbol{\Psi} = \sum_{i=0,1,2} \left(J_{2i}^{-1}(\Psi^0)^n\right)_+ \partial_i \boldsymbol{\Psi}, \tag{28}$$

$$\det J = 1, \tag{29}$$

where

$$J = \begin{pmatrix} \Psi_\lambda^0 & \Psi_\lambda^1 & \Psi_\lambda^2 \\ \Psi_x^0 & \Psi_x^1 & \Psi_x^2 \\ \Psi_y^0 & \Psi_y^1 & \Psi_y^2 \end{pmatrix}, \tag{30}$$

$\partial_0 = \partial_\lambda$, $\partial_1 = \partial_x$, $\partial_2 = \partial_y$.

The proof of (26) $\Rightarrow$ hierarchy (27)–(29) is based on the following statement.

Lemma 1. *Given identity (26), for an arbitrary first-order operator* $\hat{U}$,

$$\hat{U}\boldsymbol{\Psi} = \left(\sum_i \left(u_i^1(\lambda, \mathbf{t}^1, \mathbf{t}^2)\partial_i^1 + u_i^2(\lambda, \mathbf{t}^1, \mathbf{t}^2)\partial_i^2\right) + u^0(\lambda, \mathbf{t}^1, \mathbf{t}^2)\partial_\lambda\right)\boldsymbol{\Psi}$$

with 'plus' coefficients $\left(\left(u_i^1\right)_- = \left(u_i^2\right)_- = u_-^0 = 0\right)$, *the condition* $(\hat{U}\boldsymbol{\Psi})_+ = 0$ *(for* Ψ^1 *and* Ψ^2 *modulo the derivatives of* Ψ^0*) implies that* $\hat{U}\boldsymbol{\Psi} = 0$.

Proof. First, relation (26) implies that

$$(\det J)_- = 0,$$

and, using expansions (23)–(25), we get

$$\det J = (\det J)_+ = 1.$$

Then, using relation (26) we obtain (we use $|A|$ for $\det A$)

$$\begin{vmatrix} \hat{U}\Psi^0 & \hat{U}\Psi^1 & \hat{U}\Psi^2 \\ \Psi_x^0 & \Psi_x^1 & \Psi_x^2 \\ \Psi_y^0 & \Psi_y^1 & \Psi_y^2 \end{vmatrix}_- = \begin{vmatrix} \Psi_\lambda^0 & \Psi_\lambda^1 & \Psi_\lambda^2 \\ \hat{U}\Psi^0 & \hat{U}\Psi^1 & \hat{U}\Psi^2 \\ \Psi_y^0 & \Psi_y^1 & \Psi_y^2 \end{vmatrix}_- = \begin{vmatrix} \Psi_\lambda^0 & \Psi_\lambda^1 & \Psi_\lambda^2 \\ \Psi_x^0 & \Psi_x^1 & \Psi_x^2 \\ \hat{U}\Psi^0 & \hat{U}\Psi^1 & \hat{U}\Psi^2 \end{vmatrix}_- = 0.$$

On the other hand, condition $(\hat{U}\boldsymbol{\Psi})_+ = 0$, taking into account expansions (23), (24), (25), implies that

$$\begin{vmatrix} \hat{U}\Psi^0 & \hat{U}\Psi^1 & \hat{U}\Psi^2 \\ \Psi_x^0 & \Psi_x^1 & \Psi_x^2 \\ \Psi_y^0 & \Psi_y^1 & \Psi_y^2 \end{vmatrix}_+ = \begin{vmatrix} \Psi_\lambda^0 & \Psi_\lambda^1 & \Psi_\lambda^2 \\ \hat{U}\Psi^0 & \hat{U}\Psi^1 & \hat{U}\Psi^2 \\ \Psi_y^0 & \Psi_y^1 & \Psi_y^2 \end{vmatrix}_+ = \begin{vmatrix} \Psi_\lambda^0 & \Psi_\lambda^1 & \Psi_\lambda^2 \\ \Psi_x^0 & \Psi_x^1 & \Psi_x^2 \\ \hat{U}\Psi^0 & \hat{U}\Psi^1 & \hat{U}\Psi^2 \end{vmatrix}_+ = 0.$$

Thus, we come to the conclusion that

$$\begin{vmatrix} \hat{U}\Psi^0 & \hat{U}\Psi^1 & \hat{U}\Psi^2 \\ \Psi_x^0 & \Psi_x^1 & \Psi_x^2 \\ \Psi_y^0 & \Psi_y^1 & \Psi_y^2 \end{vmatrix} = \begin{vmatrix} \Psi_\lambda^0 & \Psi_\lambda^1 & \Psi_\lambda^2 \\ \hat{U}\Psi^0 & \hat{U}\Psi^1 & \hat{U}\Psi^2 \\ \Psi_y^0 & \Psi_y^1 & \Psi_y^2 \end{vmatrix} = \begin{vmatrix} \Psi_\lambda^0 & \Psi_\lambda^1 & \Psi_\lambda^2 \\ \Psi_x^0 & \Psi_x^1 & \Psi_x^2 \\ \hat{U}\Psi^0 & \hat{U}\Psi^1 & \hat{U}\Psi^2 \end{vmatrix} = 0.$$

This set of relations represents a linear system for $\hat{U}\boldsymbol{\Psi}$,

$$J_{\mathrm{tr}}^{-1}\hat{U}\boldsymbol{\Psi} = 0,$$

and taking into account that $\det J = 1$, the only solution to it is $\hat{U}\boldsymbol{\Psi} = 0$. □

The proof of the statement (26) $\Rightarrow$ hierarchy (27)–(29) is then straightforward, using simple relations

$$\sum_{i=0,1,2} \left(J_{1i}^{-1}(\Psi^0)^n\right)\partial_i\Psi^k = \delta_{1k}(\Psi^0)^n, \qquad \sum_{i=0,1,2} \left(J_{2i}^{-1}(\Psi^0)^n\right)\partial_i\Psi^k = \delta_{2k}(\Psi^0)^n.$$

The statement (27)–(29) $\Rightarrow$ (26) directly follows from the relation

Lemma 2.

$$\begin{vmatrix} \partial_{\tau_0}\Psi^0 & \partial_{\tau_0}\Psi^1 & \partial_{\tau_0}\Psi^2 \\ \partial_{\tau_1}\Psi^0 & \partial_{\tau_1}\Psi^1 & \partial_{\tau_1}\Psi^2 \\ \partial_{\tau_2}\Psi^0 & \partial_{\tau_2}\Psi^1 & \partial_{\tau_2}\Psi^2 \end{vmatrix} = \begin{vmatrix} V_{\tau_0+}^0 & V_{\tau_0+}^1 & V_{\tau_0+}^2 \\ V_{\tau_1+}^0 & V_{\tau_1+}^1 & V_{\tau_1+}^2 \\ V_{\tau_2+}^0 & V_{\tau_2+}^1 & V_{\tau_2+}^2 \end{vmatrix}, \tag{31}$$

where τ_0, τ_1, τ_2 is an arbitrary set of three times of the hierarchy (27)–(29), and $V^i_{\tau+}$ are the coefficients of corresponding vector fields given by the rhs of equations (27), (28),

$$\partial_{\tau_j}\mathbf{\Psi} = \sum_{i=0,1,2} V^i_{\tau_j+}\partial_i\mathbf{\Psi}.$$

Proof.

$$\det\left(\partial_{\tau_j}\Psi^k\right) = \det\left(\sum_{i=0,1,2} V^i_{\tau_j+}\partial_i\Psi^k\right) = \det\left(V^i_{\tau_j+}\right)\cdot\det J = \det\left(V^i_{\tau_j+}\right) \qquad \square$$

To complete the picture, we will also prove the compatibility of equations of the hierarchy (27)–(29).

Proposition 3. *The flows of the hierarchy (27), (28) commute and the condition* $\det J = 1$ *(29) is preserved by the dynamics.*

Proof. Let us consider, e.g., the following flows of the hierarchy:

$$\partial^1_n\mathbf{\Psi} = \sum_{i=0,1,2}\left(J^{-1}_{1i}(\Psi^0)^n\right)_+\partial_i\mathbf{\Psi} = \hat{V}^1_{n+}\mathbf{\Psi}, \tag{32}$$

$$\partial^2_m\mathbf{\Psi} = \sum_{i=0,1,2}\left(J^{-1}_{2i}(\Psi^0)^m\right)_+\partial_i\mathbf{\Psi} = \hat{V}^2_{m+}\mathbf{\Psi}, \tag{33}$$

where $\hat{V}^1_n = \sum_i\left(J^{-1}_{1i}(\Psi^0)^n\right)\partial_i$, $\hat{V}^2_m = \sum_i\left(J^{-1}_{2i}(\Psi^0)^m\right)\partial_i$ are vector fields, possessing the property

$$\hat{V}^1_n\Psi^k = \delta_{1k}(\Psi^0)^n, \tag{34}$$

$$\hat{V}^2_m\Psi^k = \delta_{2k}(\Psi^0)^m. \tag{35}$$

The condition of the compatibility of the flows (32), (33) is

$$\partial^2_m\left(\hat{V}^1_{n+}\mathbf{\Psi}\right) = \partial^1_n\left(\hat{V}^2_{m+}\mathbf{\Psi}\right). \tag{36}$$

First, in a standard way

$$\partial^2_m\left(\hat{V}^1_{n+}\mathbf{\Psi}\right) - \partial^1_n\left(\hat{V}^2_{m+}\mathbf{\Psi}\right) = \left(\partial^2_m\hat{V}^1_{n+} - \partial^1_n\hat{V}^2_{m+} + \left[\hat{V}^1_{n+}, \hat{V}^2_{m+}\right]\right)\mathbf{\Psi}.$$

Relation $\hat{U}\mathbf{\Psi} = 0$ for some vector field $\hat{U}$ may be considered as a homogeneous linear system for the coefficients of the vector field $\hat{U}$ with the matrix J (29). Thus $\hat{U}\mathbf{\Psi} = 0 \Rightarrow \hat{U} = 0$, and condition (36) is equivalent to the usual form of compatibility condition

$$\partial^2_m\hat{V}^1_{n+} - \partial^1_n\hat{V}^2_{m+} + \left[\hat{V}^1_{n+}, \hat{V}^2_{m+}\right] = 0. \tag{37}$$

On the other hand, using (34), (35), we get

$$\partial^2_m\left(\hat{V}^1_{n+}\mathbf{\Psi}\right) - \partial^1_n\left(\hat{V}^2_{m+}\mathbf{\Psi}\right) = -\left(\partial^2_m\hat{V}^1_{n-} - \partial^1_n\hat{V}^2_{m-} - \left[\hat{V}^1_{n-}, \hat{V}^2_{m-}\right]\right)\mathbf{\Psi}.$$

Then

$$\begin{aligned}
&\hat{W}\mathbf{\Psi} = (\hat{W}_+ + \hat{W}_-)\mathbf{\Psi} = 0,\\
&\hat{W}_+ = \left(\partial^2_m\hat{V}^1_{n+} - \partial^1_n\hat{V}^2_{m+} + \left[\hat{V}^1_{n+}, \hat{V}^2_{m+}\right]\right)\\
&\hat{W}_- = \left(\partial^2_m\hat{V}^1_{n-} - \partial^1_n\hat{V}^2_{m-} - \left[\hat{V}^1_{n-}, \hat{V}^2_{m-}\right]\right),
\end{aligned}$$

and we come to the conclusion that $\hat{W} = 0$, so, evidently, $\hat{W}_+ = \hat{W}_- = 0$, that proves the compatibility condition (37).

The conservation of volume (29) by some flow $\partial_\tau \boldsymbol{\Psi} = \sum_i V^i_{\tau+}\partial_i \boldsymbol{\Psi}$ is equivalent to the zero divergence condition

$$\sum_i \partial_i V^i_{\tau+} = 0, \tag{38}$$

which can be checked directly starting from the definition of the hierarchy. Indeed,

$$J^{-1}_{1i} = -\sum_{j,k} \epsilon_{ijk}\partial_j \Psi^0 \partial_k \Psi^2, \qquad J^{-1}_{2i} = \sum_{j,k} \epsilon_{ijk}\partial_j \Psi^0 \partial_k \Psi^1,$$

and the calculation of divergence (38) for any of the flows (27), (28) leads to the appearance of symmetry for a pair of indices under summation with a completely antisymmetric symbol ϵ_{ijk}, giving zero as the result. □

In a more explicit form, the Dunajski equation hierarchy (27), (28) can be written as

$$\begin{aligned}
\partial^1_n \boldsymbol{\Psi} = &+ \left((\Psi^0)^n \begin{vmatrix} \Psi^0_\lambda & \Psi^2_\lambda \\ \Psi^0_y & \Psi^2_y \end{vmatrix} \right)_+ \partial_x \boldsymbol{\Psi} - \left((\Psi^0)^n \begin{vmatrix} \Psi^0_\lambda & \Psi^2_\lambda \\ \Psi^0_x & \Psi^2_x \end{vmatrix} \right)_+ \partial_y \boldsymbol{\Psi} \\
&- \left((\Psi^0)^n \begin{vmatrix} \Psi^0_x & \Psi^2_x \\ \Psi^0_y & \Psi^2_y \end{vmatrix} \right)_+ \partial_\lambda \boldsymbol{\Psi},
\end{aligned} \tag{39}$$

$$\begin{aligned}
\partial^2_n \boldsymbol{\Psi} = &- \left((\Psi^0)^n \begin{vmatrix} \Psi^0_\lambda & \Psi^1_\lambda \\ \Psi^0_y & \Psi^1_y \end{vmatrix} \right)_+ \partial_x \boldsymbol{\Psi} + \left((\Psi^0)^n \begin{vmatrix} \Psi^0_\lambda & \Psi^1_\lambda \\ \Psi^0_x & \Psi^1_x \end{vmatrix} \right)_+ \partial_y \boldsymbol{\Psi} \\
&+ \left((\Psi^0)^n \begin{vmatrix} \Psi^0_x & \Psi^1_x \\ \Psi^0_y & \Psi^1_y \end{vmatrix} \right)_+ \partial_\lambda \boldsymbol{\Psi}
\end{aligned} \tag{40}$$

(plus equation (29)). It is easy to check that for $\Psi^0 = \lambda$ the Dunajski equation hierarchy reduces to heavenly equation hierarchy [10, 11], while for $\Psi^2 = y$ it reduces to the dispersionless KP hierarchy. The first two flows of the hierarchy (39), (40) read

$$\partial^1_1 \boldsymbol{\Psi} = (u_y \partial_x - u_x \partial_y + \lambda \partial_x - f_x \partial_\lambda)\boldsymbol{\Psi}, \tag{41}$$

$$\partial^2_1 \boldsymbol{\Psi} = (v_x \partial_y - v_y \partial_x + \lambda \partial_y - f_y \partial_\lambda)\boldsymbol{\Psi}, \tag{42}$$

which yields the Lax pair (4) after the identification $z = -t^1_1$, $w = t^2_1$. Here

$$u = \Psi^2_1, \qquad v = \Psi^1_1, \qquad f = \Psi^0_1,$$

and condition $\det J = 1$ (29) implies that $u_y + v_x = 0$.

The second flows can be written in the form

$$\partial^1_2 \boldsymbol{\Psi} = \big(\lambda \partial^1_1 + f \partial_x - \big(\partial^1_1 f\big)\partial_\lambda\big)\boldsymbol{\Psi}, \tag{43}$$

$$\partial^2_2 \boldsymbol{\Psi} = \big(\lambda \partial^2_1 + f \partial_y - \big(\partial^2_1 f\big)\partial_\lambda\big)\boldsymbol{\Psi}. \tag{44}$$

To write down the vector field more explicitly, one should use the first flows (41), (42). Commutation relations for any pair of the flows give an equation for Θ, f (with different set of times).

3.1. Related hierarchies

Formula (29) defines a reduction for equations (27), (28). The general hierarchy in the unreduced case is given by equations (27), (28),

$$\partial_n^1 \boldsymbol{\Psi} = \sum_{i=0,1,2} \left(J_{1i}^{-1}(\Psi^0)^n\right)_+ \partial_i \boldsymbol{\Psi}, \tag{45}$$

$$\partial_n^2 \boldsymbol{\Psi} = \sum_{i=0,1,2} \left(J_{2i}^{-1}(\Psi^0)^n\right)_+ \partial_i \boldsymbol{\Psi}, \tag{46}$$

and the analogue of relation (26) is

$$((\det J)^{-1}\,\mathrm{d}\Psi^0 \wedge \mathrm{d}\Psi^1 \wedge \mathrm{d}\Psi^2)_- = 0. \tag{47}$$

For the unreduced hierarchy the propositions formulated above and their proofs are completely analogous. The hierarchy may also be considered for an arbitrary number of components Ψ^0, Ψ^i.

The two-component case of relation (47)

$$((\det J)^{-1}\,\mathrm{d}\Psi^0 \wedge \mathrm{d}\Psi^1)_- = 0 \tag{48}$$

and the corresponding equations (45) define a hierarchy for the system introduced in [15] (dispersionless KP minus area conservation). If $\det J = 1$, relation (48) defines the dispersionless KP hierarchy.

A special subclass of the hierarchies of the type (47), (26) is singled out by the condition $\Psi^0 = \lambda$. In this case (26) is transformed to the relation

$$(\mathrm{d}\lambda \wedge \mathrm{d}\Psi^1 \wedge \mathrm{d}\Psi^2)_- = 0,$$

which is equivalent to

$$(\tilde{\mathrm{d}}\Psi^1 \wedge \tilde{\mathrm{d}}\Psi^2)_- = 0, \tag{49}$$

where the differential $\tilde{\mathrm{d}}$ takes into account only times (and not a spectral variable). As is known, relation (49) defines the Plebanski second heavenly equation hierarchy [10, 11] (see also [17, 19]). A two-component case of (47) (relation (48)) under the condition $\Psi^0 = \lambda$ reduces to

$$\left(\left(\Psi_x^1\right)^{-1} \tilde{\mathrm{d}}\Psi^1\right)_- = 0.$$

This relation defines a hierarchy considered in [21], the corresponding equations (45) define a system of (positive) flows of this hierarchy.

4. Summary

The Dunajski equation, considered as an integrable system, generalizes both the dispersionless KP and Plebański second heavenly equations. The dressing scheme for this equation is based on the nonlinear Riemann problem (6) with a volume conservation condition (11). Special solutions can be constructed in terms of implicit functions (equations (17), (18) or (19), (20)). The hierarchy corresponding to the Dunajski equation is characterized by relation (26), where the differential includes both times and a spectral variable, and it can be written in the Lax–Sato form (27)–(29). The Dunajski equation hierarchy demonstrates characteristic features of the general hierarchy connected with the commutation of vector fields, and opens a way to understanding of the structure of the general hierarchy, which can be defined by the multicomponent relation of the type (26) (or (47)) with an equivalent Lax–Sato form (27)–(29) (or (45), (46)). This hierarchy contains several known integrable models as special cases (subsection 3.1).

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